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ON NEW REPRESENTATIONS OF HIKAMI'S MOCK THETA FUNCTIONS AND MOCK THETA FUNCTION OF ORDER THREE, FIVE AND SEVEN

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ABSTRACT: In this paper, we have established certain new representations of Mock theta functions due to Ramanujan and Hikami respectively by making use of identity duo to Srivastava, A. K.
KEYWORDS: Mock theta functions, Hikami Mock theta functions, identity order.

1. INTRODUCTION

Ramanujan made profound contributions in the field of $q-$series and in his lost notebook more attention had been paid to hypergeometric series and basic hypergeometric series. Ramanujan rediscovered a number of classical formulae including those attached to the names of Gauss, Kummer, Dixon, Dougall. Although Ramanujan left no clue of how he discovered these fascinating formulae and theorems. Ramanujan in his last communication to Hardy, quoted about certain Mock theta functions of order three, five and seven in his and later on Watson, G. N. [14,15], Andrews, G. E. [5] and Agarwal, R. P. [1] contributed significantly to the theory of Mock theta functions. The general identities introduced by Andrews G. E. [4,5] related with Mock theta functions was a very general class of basic hypergeometric transformation and it was shown by Agarwal, R. P. [2,3] in his work based on Mock theta functions. Srivastava, A. K. [11] made an attempt about partial Mock theta functions, due to this process interrelationship came in picture as partial Mock theta functions plays a very crucial role regarding computation of new Mock theta functions. Srivastava Pankaj [12] introduced compact form of representation of all Mock theta functions by making use of identity due to Jain, V. K. and Srivastava, H. M. [6] later on Srivastava Pankaj and Wahidi, A. J. [13] studied basic hypergeometric structure of Hikami's Mock theta functions with some its properties. Recently M. Pathak and Srivastava Pankaj [9] established new representations of complete Mock theta functions of order eight. The present research article deals with new representation of Mock theta functions introduced by Ramanujan and Hikami [7,8] respectively by making use of identity established by Srivastava, A. K.

2. Notations and Definitions

For real or complex q, $|q|<1$, let

$$[a;q]_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\qquad [a;q]_0=1 \tag{2.1}$$

$$[a;q]_\infty=\prod_{n=0}^{\infty}(1-aq^n) \tag{2.2}$$

$$[a;q]_n=\frac{[a;q]_\infty}{[aq^n;q]_\infty} \tag{2.3}$$

Basic hypergeometric series is defined as:

$${}_r\phi_s\left[\begin{matrix}a_1,a_2,\cdots,a_r;q;z\\ b_1,b_2,\cdots,b_s;q^{\lambda}\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{[a_1,a_2,\ldots,a_r;q]_n z^n q^{\lambda n(n-1)/2}}{[q,b_1,b_2,\ldots,b_s;q]_n},$$

Convergent for $|q|<1$, $|z|<\infty$ if $\lambda>0$ and $\max(|q|,|z|)<1$ if $\lambda=0$.

Also, $$[a_1,a_2,\ldots,a_r;q]_n=[a_1;q]_n[a_2;q]_n\ldots[a_r;q]_n. \tag{2.4}$$

A truncated basic hypergeometric series is defined as:

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \cdots, a_r; q; z \\ b_1, b_2, \cdots, b_s; q^{\lambda} \end{matrix}\right]_N = \sum_{n=0}^{N} \frac{[a_1, a_2, \ldots, a_r; q]_n z^n q^{\lambda n(n-1)/2}}{[q, b_1, b_2, \ldots, b_s; q]_n}. \qquad (2.5) $$

We shall make use of following identity in our analysis:

$$ A(q)\sum_{m=0}^{\infty} B_m(q)\sum_{r=0}^{m}\alpha_r + C_{\infty}(q)\sum_{m=0}^{\infty}\alpha_m = \sum_{m=0}^{\infty} c_m(q)\alpha_m, $$

where

$$ A(q) = \frac{(aq-e)(e-bq)}{(q-e)(e-abq)}, \qquad B_m(q) = \frac{[a,b;q]_m q^m}{[e, abq/e; q]_m}, $$

$$ c_m(q) = \frac{[a,b;q]_m}{[e/q, abq/e; q]_m} \quad \text{and} \quad c_{\infty}(q) = \frac{[a,b;q]_{\infty}}{[e/q, abq/e; q]_{\infty}} \qquad (2.6) $$

a) Mock theta functions of order three are:

$$ f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q;q]_n^2}, \qquad \phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-q^2;q^2]_n}, $$

$$ \psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q^2]_n}, \qquad \chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[-\omega q, -\omega^2 q; q]_n}, $$

$$ \omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}^2}, \qquad v(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{[-q;q^2]_{n+1}}, $$

$$ \rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[\omega q, \omega^2 q; q^2]_{n+1}}. \qquad (2.7) $$

Partial Mock theta functions of order three:

$$ f_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-q;q]_n^2}, \qquad \phi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-q^2;q^2]_n}, $$

$$ \psi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[q;q^2]_n}, \qquad \chi_N(q) = \sum_{n=0}^{N} \frac{q^{n^2}}{[-\omega q, -\omega^2 q; q]_n}, \qquad (2.8) $$

$$ \omega_N(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{[q;q^2]_{n+1}^2}, \qquad v_N(q) = \sum_{n=0}^{N} \frac{q^{n(n+1)}}{[-q;q^2]_{n+1}}, $$

$$ \rho_N(q) = \sum_{n=0}^{N} \frac{q^{2n(n+1)}}{[\omega q, \omega^2 q; q^2]_{n+1}}. $$

b) Mock theta functions of order five are:

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n}, \qquad \phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}[-q;q^2]_n,$$

$$\psi_0(q)=\sum_{n=0}^{\infty}[-q;q]_n q^{(n+1)(n+2)/2}, \qquad F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{[q;q^2]_n},$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n[q;q]_n}{[q;q]_{2n}},$$

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[-q;q]_n}, \qquad \phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}[-q;q^2]_n,$$

$$\psi_1(q)=\sum_{n=0}^{\infty}[-q;q]_n q^{n(n+1)/2}, \qquad F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{[q;q^2]_{n+1}},$$

$$\chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n[q;q]_n}{[q;q]_{2n+1}}. \tag{2.9}$$

Partial Mock theta functions of order five:

$$f_{0,N}(q)=\sum_{n=0}^{N}\frac{q^{n^2}}{[-q;q]_n}, \qquad \phi_{0,N}(q)=\sum_{n=0}^{N}q^{n^2}[-q;q^2]_n,$$

$$\psi_{0,N}(q)=\sum_{n=0}^{N}[-q;q]_n q^{(n+1)(n+2)/2}, \qquad F_{0,N}(q)=\sum_{n=0}^{N}\frac{q^{2n^2}}{[q;q^2]_n},$$

$$\chi_{0,N}(q)=\sum_{n=0}^{N}\frac{q^n[q;q]_n}{[q;q]_{2n}},$$

$$f_{1,N}(q)=\sum_{n=0}^{N}\frac{q^{n(n+1)}}{[-q;q]_n}, \qquad \phi_{1,N}(q)=\sum_{n=0}^{N}q^{(n+1)^2}[-q;q^2]_n,$$

$$\psi_{1,N}(q)=\sum_{n=0}^{N}[-q;q]_n q^{n(n+1)/2}, \qquad F_{1,N}(q)=\sum_{n=0}^{N}\frac{q^{2n(n+1)}}{[q;q^2]_{n+1}},$$

$$\chi_{1,N}(q)=\sum_{n=0}^{N}\frac{q^n[q;q]_n}{[q;q]_{2n+1}}. \tag{2.10}$$

d) Mock theta functions of order seven are:

$$\Im_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}[q;q]}{[q;q]_{2n}}, \qquad \Im_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}},$$

$$\Im_2(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}[q;q]_n}{[q;q]_{2n+1}} \tag{2.11}$$

Partial Mock theta functions of order seven:

$$\Im_{0,N}(q)=\sum_{n=0}^{N}\frac{q^{n^2}[q;q]}{[q;q]_{2n}}, \qquad \Im_{1,N}(q)=\sum_{n=0}^{N}\frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}},$$

$$\Im_{2,N}(q)=\sum_{n=0}^{N}\frac{q^{n^2+n}[q;q]_n}{[q;q]_{2n+1}}. \tag{2.12}$$

e) Hikami Mock theta functions of order two, four and eight:

$$D_5(q)=\sum_{n=0}^{\infty}\frac{q^n[-q;q]_n}{[q;q^2]_{n+1}},\qquad D_6(q)=\sum_{n=0}^{\infty}\frac{q^n[-q^2;q^2]_n}{[q^{n+1};q]_{n+1}},$$
$$I_{12}(q)=\sum_{n=0}^{\infty}\frac{q^{2n}[-q;q^2]_n}{[q^{n+1};q]_{n+1}},\qquad I_{13}(q)=\sum_{n=0}^{\infty}\frac{q^n[-q;q^2]_n}{[q^{n+1};q]_{n+1}} \tag{2.13}$$

Partial Hikami Mock theta function

$$D_{5,N}(q)=\sum_{n=0}^{N}\frac{q^n[-q;q]_n}{[q;q^2]_{n+1}},\qquad D_{6,N}(q)=\sum_{n=0}^{N}\frac{q^n[-q^2;q^2]_n}{[q^{n+1};q]_{n+1}},$$
$$I_{12,N}(q)=\sum_{n=0}^{N}\frac{q^{2n}[-q;q^2]_n}{[q^{n+1};q]_{n+1}},\qquad I_{13,N}(q)=\sum_{n=0}^{N}\frac{q^n[-q;q^2]_n}{[q^{n+1};q]_{n+1}}. \tag{2.14}$$

3. MAIN RESULTS

In this section we shall establish new representations involving partial Mock theta functions of order three, five and seven. Also, partial Hikami Mock theta functions of order two, four and eight respectively.

a) Representation of Mock theta functions of order three:

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}f(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,-q,-q;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)f_m(q) \tag{3.1}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\phi(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,iq,-iq;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\,\phi_m(q) \tag{3.2}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\psi(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,q^{1/2},-q^{1/2};q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\psi_m(q) \tag{3.3}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\chi(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,-\omega q,-\omega^2q;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\,\chi_m(q) \tag{3.4}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\omega(q)=\frac{1}{(1-q)^2}\,{}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,q^2,q^2;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\,\omega_m(q) \tag{3.5}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\upsilon(q)=\frac{1}{(1+q)}\,{}_2\phi_3\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,-q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\upsilon_m(q) \tag{3.6}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\rho(q)=\frac{1}{(1-\omega q)(1-\omega^2q)}\,{}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,\omega q^3\omega^2q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\,\rho_m(q) \tag{3.7}$$

b) ***Representation of Mock theta functions of order five:***

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}f_0(q)={}_2\phi_3\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,-q;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\,f_{0,m}(q) \tag{3.8}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\phi_0(q)={}_4\phi_2\left[\begin{matrix}a,b,iq^{1/2},-iq^{1/2};q;q^2\\ e/q,abq/e;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\phi_{0,m}(q) \tag{3.9}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\psi_0(q)=q^2\,{}_3\phi_2\left[\begin{matrix}a,b,-q;q;q^2\\ e/q,abq/e;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\psi_{0,m}(q) \tag{3.10}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}F_0(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,q^{1/2},-q^{1/2};q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)F_{0,m}(q) \tag{3.11}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\chi_0(q)={}_2\phi_4\left[\begin{matrix}a,b;q;q\\ e/q,abq/e,q,q^2;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\chi_{0,m}(q) \tag{3.12}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}f_1(q)={}_2\phi_3\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,-q;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)f_{1,m}(q) \tag{3.13}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\phi_1(q)=q\,{}_4\phi_2\left[\begin{matrix}a,b,iq^{1/2},-iq^{1/2};q;q^2\\ e/q,abq/e;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\phi_{1,m}(q) \tag{3.14}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\psi_1(q)={}_3\phi_2\left[\begin{matrix}a,b,-q;q;q^2\\ e/q,abq/e;q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\psi_{1,m}(q) \tag{3.15}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}F_1(q)=\frac{1}{1-q}\,{}_2\phi_3\left[\begin{matrix}a,b;q;q^2\\ e/q,abq/e,q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)F_{1,m}(q) \tag{3.16}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\chi_1(q)={}_3\phi_4\left[\begin{matrix}a,b,q;q;q\\ e/q,abq/e,q^2,q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\chi_{1,m}(q) \tag{3.17}$$

c) Representation of Mock theta functions of order seven:

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\Im_0(q)={}_3\phi_4\left[\begin{matrix}a,b,q;q;q^2\\ e/q,abq/e,q,q^2;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\Im_{0,m}(q) \tag{3.18}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\Im_1(q)=q\,{}_3\phi_4\left[\begin{matrix}a,b,q;q;q^2\\ e/q,abq/e,q^2,q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\Im_{1,m}(q) \tag{3.19}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}\Im_2(q)={}_3\phi_4\left[\begin{matrix}a,b;q;q;q^2\\ e/q,abq/e,q^2,q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)\Im_{2,m}(q) \tag{3.20}$$

d) Representations of Hikami's Mock theta functions of order two, four and eight:

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}D_5(q)=\frac{1}{1-q}\,{}_3\phi_4\left[\begin{matrix}a,b,-q;q;q\\ e/q,abq/e,q^3;q^2\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)D_{5,m}(q) \tag{3.21}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty}D_6(q)={}_4\phi_5\left[\begin{matrix}a,b,iq,-iq;q;q\\ e/q,abq/e,-q,q^{3/2},-q^{3/2};q\end{matrix}\right]-A(q)\sum_{m=0}^{\infty}B_m(q)D_{6,m}(q) \tag{3.22}$$

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty} I_{12}(q) = {}_4\phi_5\left[\begin{matrix} a,b,iq^{1/2},-iq^{1/2};q;q^2 \\ e/q,abq/e,-q,q^{3/2},-q^{3/2};q \end{matrix}\right] - A(q)\sum_{m=0}^{\infty} B_m(q)\, I_{12,m}(q)$$

(3.23)

$$\frac{[a,b;q]_\infty}{[e/q,abq/e;q]_\infty} I_{13}(q) = {}_4\phi_5\left[\begin{matrix} a,b,iq^{1/2},-iq^{1/2};q;q \\ e/q,abq/e,-q,q^{3/2},-q^{3/2};q \end{matrix}\right] - A(q)\sum_{m=0}^{\infty} B_m(q)\, I_{13,m}(q)$$

(3.24)

4. PROOF OF MAIN RESULTS

As an illustration, we shall prove result (3.1)

Taking $\alpha_m = \frac{q^{m^2}}{[-q;q]_m^2}$ and putting the values of $c_m(q)$ and $c_\infty(q)$ in (2.6) we get,

$$A(q)\sum_{m=0}^{\infty} B_m(q)\, f_m(q) + \frac{(a,b;q)_\infty}{(e/q,abq/e;q)_\infty} f(q) = \sum_{m=0}^{\infty} \frac{(a,b;q)_m}{(e/q,abq/e;q)_m} \frac{q^{m^2}}{[-q;q]_m^2}, \quad (4.1)$$

and after simplification we get, the result (3.1). Similarly, suitable selections of α_m and making use of respective identity one can establish the results (3.2) to (3.24).

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